Model Reduction of Mechanical Systems

Ajmal Yousuff* and Mary Breida†
Drexel University, Philadelphia, Pennsylvania 19104

Introduction

THIS Note deals with obtaining a reduced model of a stable mechanical system with proportional damping. Such systems can be conveniently represented in modal coordinates. The popular scheme modal cost analysis (MCA), given in Ref. 1, offers a simple means of identifying dominant modes for retention in the reduced model. In MCA the dominance is measured via the modal costs. Although this measure leads to simple computations, it does not exactly reflect the more appropriate model error, which is the L_2 norm of the output error between the full and the reduced models. Normally, the model error is computed *after* the reduced model is obtained, since it is believed that, in general, the model error cannot be easily computed a priori. The main thrust of this Note is to point out that the model error can also be calculated a priori, just as easily as the modal costs. Hence, the model error itself can be used to determine the dominant modes. Moreover, the simplicity of the computations do not presume any special properties of the system, such as small damping and orthogonal symmetry. The development presented herein can be seen as a specialization of that in Ref. 2 to mechanical systems.

Problem Formulation

Consider a time-invariant mechanical system, described in its physical coordinates q, given in the following:

$$\mathcal{M}\ddot{q}(t) + \mathcal{D}\dot{q}(t) + \mathcal{K}q(t) = \mathcal{B}u(t)$$
$$y(t) = \mathcal{C}q(t) + \underline{\mathcal{C}}\dot{q}(t) \tag{1}$$

where $q \in \boldsymbol{R}^n$, $u \in \boldsymbol{R}^m$, and $y \in \boldsymbol{R}^k$, and u is assumed to be a Gaussian white noise process with unit intensity. Both the mass matrix $\mathcal{M}$ and the stiffness matrix $\mathcal{K}$ are assumed to be symmetric and positive definite. The dissipation matrix $\mathcal{D}$ is assumed to arise from proportional damping (not necessarily small) such that the above system is asymptotically stable. Under these assumptions model (1) can be equivalently expressed in its modal coordinates as

$$\ddot{\eta}_i + 2\zeta_i\omega_i\dot{\eta}_i + \omega_i^2\eta_i = b_i^Tu; \qquad i = 1, 2, \ldots, n \tag{2a}$$

$$y = \sum_{i=1}^{n} (c_i\eta_i + \underline{c}_i\dot{\eta}_i) \tag{2b}$$

where ω_i and ζ_i are the usual ith natural frequency and the corresponding damping ratio, respectively.

When n is large, one is faced with the problem of model reduction to facilitate a subsequent analysis and control design. Quite often, a reduced model of the system [(2a) and (2b)] is obtained by retaining some r modes of the total n modes—a process we call a "modal reduction." To produce an acceptable reduced model, the question that needs to be answered is "which r of the n modes should be retained?" Using the notation N_n for the integer set $\{1, 2, \ldots, n\}$, the issue in modal reduction is to identify an r-element "reduction set" $N_{\text{red}} \subset N_n$, such that the following model,

$$\ddot{\eta}_i + 2\zeta_i\omega_i\dot{\eta}_i + \omega_i^2\eta_i = b_i^Tu; \qquad i \in N_{\text{red}} \tag{3a}$$

$$y_r = \sum_{i \in N_{\text{red}}} (c_i\eta_i + \underline{c}_i\dot{\eta}_i) \tag{3b}$$

would be an acceptable reduced model. Defining the error $e(t)$ between the full model [(2a) and (2b)] and the reduced model [(3a) and (3b)] as $e(t) = y(t) - y_r(t)$, an acceptable reduced model would minimize the model error $\delta\mathcal{V}$ defined as follows:

$$\delta\mathcal{V} = \lim_{t\to\infty} \mathcal{E}\left\{\frac{1}{t}\int_0^t e^T(\sigma)e(\sigma)\,\mathrm{d}\sigma\right\} \tag{4}$$

Dominant Modes by Modal Cost Analysis

For an arbitrary system, not necessarily a mechanical system, the model error $\delta\mathcal{V}$ associated with its reduced model is normally computed after the reduced model is obtained. Consequently, obtaining an acceptable reduced model for such systems becomes an iterative process. To simplify the reduction process, methods have been proposed to employ different criteria instead of Eq. (4). One such method is the component cost analysis (CCA) of Skelton and Yousuff.[3] The CCA attempts to use the cost error $\Delta\mathcal{V}$, defined in the following, as its criterion for model reduction:

$$\Delta\mathcal{V} = \mathcal{V} - \mathcal{V}_r \tag{5a}$$

where

$$\mathcal{V} = \lim_{t\to\infty} \mathcal{E}\left\{\frac{1}{t}\int_0^t y^T(\sigma)y(\sigma)\,\mathrm{d}\sigma\right\} \tag{5b}$$

$$\mathcal{V}_r = \lim_{t\to\infty} \mathcal{E}\left\{\frac{1}{t}\int_0^t y_r^T(\sigma)y_r(\sigma)\,\mathrm{d}\sigma\right\} \tag{5c}$$

Now the reduced models, which are optimal in the sense of minimizing the model error $\delta\mathcal{V}$, are known to satisfy the following orthogonal property[4]:

$$\langle e, y_r\rangle = \lim_{t\to\infty} \mathcal{E}\left\{\frac{1}{t}\int_0^t e^T(\sigma)y_r(\sigma)\,\mathrm{d}\sigma\right\} = 0 \tag{6}$$

Since the model error $\delta\mathcal{V}$ can be expressed in terms of the cost error $\Delta\mathcal{V}$ and the inner product $\langle e, y_r\rangle$, as given in the following,

$$\delta\mathcal{V} = \Delta\mathcal{V} - 2\langle e, y_r\rangle \tag{7}$$

it follows that the cost error is an appropriate criteria to use, provided the resulting reduced model is known to be near optimal, if not optimal. Although the computation of $\Delta\mathcal{V}$ is simpler than that of $\delta\mathcal{V}$, it still requires the availability of the reduced model. In view of this, the CCA employs the predicted cost error $\widehat{\Delta\mathcal{V}}$, defined in the following, for its model reduction decisions:

$$\widehat{\Delta\mathcal{V}} = \mathcal{V} - \hat{\mathcal{V}}_r \tag{8}$$

where $\hat{\mathcal{V}}_r$ is a prediction of $\mathcal{V}_r$. With x_i defined as the (first-order) state representing the ith component, $\hat{\mathcal{V}}_r$ is computed according to

$$\hat{\mathcal{V}}_r = \sum_{i \in N_{\text{red}}} \hat{\mathcal{V}}_i \tag{9a}$$

and where $\hat{\mathcal{V}}_i$, the ith component cost, is defined as[3]

$$\hat{\mathcal{V}}_i = \frac{1}{2}\lim_{t\to\infty} \mathcal{E}\left\{\frac{1}{t}\int_0^t \frac{\partial\left[y^T(\sigma)y(\sigma)\right]}{\partial x_i(\sigma)}x_i(\sigma)\,\mathrm{d}\sigma\right\} \tag{9b}$$

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*Associate Professor, Department of Mechanical Engineering and Mechanics. Member AIAA.

†Graduate Student, Department of Mechanical Engineering and Mechanics. Student Member AIAA.

Since the calculation of the component costs requires only the full model, the predicted cost error $\widehat{\Delta \mathcal{V}}$ can be computed before a reduced model is obtained.

The application of the CCA to models represented in modal coordinates, such as in Eqs. (2a) and (2b), is called the modal cost analysis (MCA).[1] Therefore, modal reduction by MCA chooses an N_{red} such that the predicted cost error $\widehat{\Delta \mathcal{V}}$ is minimized. Except under special cases, there is no guarantee that the predicted cost error $\widehat{\Delta \mathcal{V}}$ equals the model error $\delta \mathcal{V}$. It turns out that for any choice of N_{red} the corresponding $\delta \mathcal{V}$ can be computed just as easily as $\widehat{\Delta \mathcal{V}}$ is computed. Hence, one may use the more appropriate criterion $\delta \mathcal{V}$ for choosing N_{red}. These details are developed in the following.

Closed-Form Expression for Model Error

Provided all the modes are observable and controllable, truncation of any mode would affect the output y. To determine such an effect, define

$$\psi_i = c_i \eta_i + \underline{c}_i \dot{\eta}_i \qquad \text{so that} \qquad y = \sum_{i \in N_n} \psi_i \tag{10}$$

and call ψ_i the ith modal output. By defining the truncation set $N_{\text{tru}} = N_n - N_{\text{red}}$, the following expressions result:

$$e = \sum_{i \in N_{\text{tru}}} \psi_i \tag{11a}$$

$$\delta \mathcal{V} = \sum_{i \in N_{\text{tru}}} \left[\sum_{j \in N_{\text{tru}}} \Psi_{ij} \right] \tag{11b}$$

where

$$\Psi_{ij} = \lim_{t \to \infty} \mathcal{E} \left\{ \frac{1}{t} \int_0^t \psi_i^T(\sigma) \psi_j(\sigma) \, \mathrm{d}\sigma \right\}; \qquad i,j \in N_n \tag{11c}$$

In the spirit of CCA, call Ψ_{ij} the cost correlation between the modes i and j. Note that Ψ_{ij} measures the correlation between the ith and the jth modal outputs and depends on only the ith and the jth modal data. The expression for computing Ψ_{ij} for all $i, j \in N_n$ is given in the next section. Given the modal data, these cost correlations can be computed for all of the modes, and the model error $\delta \mathcal{V}$ can be determined for any r-element reduction set $N_{\text{red}} \subset N_n$ from the expression (11b). Notice that the model error can be computed a priori by simple summation of appropriate Ψ_{ij}.

Clearly, there are $[n!/\{r!(n-r)!\}]$ number of possible reduction sets, of which one will yield the smallest model error. To facilitate the selection of the best reduction set, construct an (n,n) cost correlation matrix Ψ whose (i,j) element equals Ψ_{ij} and define an operation $\Sigma\Sigma : \boldsymbol{R}^{(n,m)} \to \boldsymbol{R}$ for arbitrary n and m by

$$\Sigma\Sigma(Z) = \sum_{i=1}^{n} \sum_{j=1}^{m} Z_{ij}$$

Then the cost correlation matrix possesses the following properties for any N_{red}:

Property 1:

$$\delta \mathcal{V} = \Sigma\Sigma(\Psi_{\text{tru}}) \tag{12a}$$

Property 2:

$$\mathcal{V} = \Sigma\Sigma(\Psi) \tag{12b}$$

Property 3:

$$\mathcal{V}_r = \Sigma\Sigma(\Psi_{\text{red}}) \tag{12c}$$

Property 4:

$$\langle e, y_r \rangle = \Sigma\Sigma(\Psi_{\text{retr}}) \tag{12d}$$

Property 5:

$$\hat{\mathcal{V}}_i = \Sigma\Sigma(\Psi_i) \tag{12e}$$

where

$$\Psi_{\text{tru}} = [\Psi_{ij};\ i,j \in N_{\text{tru}}] \tag{13a}$$

an $(n-r, n-r)$ submatrix of Ψ,

$$\Psi_{\text{red}} = [\Psi_{ij};\ i,j \in N_{\text{red}}] \tag{13b}$$

an (r,r) submatrix of Ψ,

$$\Psi_{\text{retr}} = [\Psi_{ij};\ i \in N_{\text{red}},\ \ j \in N_{\text{tru}}] \tag{13c}$$

an $(r, n-r)$ submatrix of Ψ, and

$$\Psi_i = [\Psi_{ij};\ j \in N_n] \tag{13d}$$

an $(1,n)$ matrix (i.e., ith row) for all i.

Notice that the fourth property of the cost correlation matrix allows one to determine a suboptimality index[5] associated with the reduced model. This index can be used to determine whether the reduced model could be improved by further optimizations. It measures how closely the orthogonality condition (6) is satisfied. The second and third properties provide means of computing the quadratic cost functions $\mathcal{V}$ and $\mathcal{V}_r$ associated with the full and the reduced models, respectively. Moreover, the three errors (namely, the model error $\delta \mathcal{V}$, the cost error $\Delta \mathcal{V}$, and the predicted cost error $\widehat{\Delta \mathcal{V}}$) are related through property (12d) as follows:

$$\Delta \mathcal{V} = \delta \mathcal{V} + 2\Sigma\Sigma(\Psi_{\text{retr}}) \tag{14a}$$

$$\widehat{\Delta \mathcal{V}} = \delta \mathcal{V} + \Sigma\Sigma(\Psi_{\text{retr}}) \tag{14b}$$

Hence, unless $\Sigma\Sigma(\Psi_{\text{retr}}) \approx 0$, modes determined as dominant based on either the cost error or the predicted cost error may not yield the smallest model error. Since the expression (11b) is available for the computation of the model error for any choice of N_{red}, dominance of modes can be established based on $\delta \mathcal{V}$ itself. The disadvantage, however, is in having to compute $[(n-r)^2 - 1]$ additions $[n!/\{r!(n-r)!\}]$ times.

The expression for computing Ψ_{ij} for all $i,j \in N_n$ is given in the following:

$$\Psi_{ij} = X_{ij}^{11} \left[c_i^T c_j - \frac{\Delta_{ij}}{\delta_{ij}} (\underline{c}_i^T c_j - c_i^T \underline{c}_j) - \frac{\Delta_{ij}^2}{\delta_{ij}^2} \underline{c}_i^T \underline{c}_j \right] + \frac{b_i^T b_j}{\delta_{ij}} \underline{c}_i^T \underline{c}_j \tag{15a}$$

where

$$X_{ij}^{11} = \frac{(b_i^T b_j)\delta_{ij}}{\Delta_{ij}^2 + \omega_i^2 \delta_{ij}^2 - 2\zeta_i \omega_i \delta_{ij} \Delta_{ij}} \tag{15b}$$

$$\Delta_{ij} = \omega_i^2 - \omega_j^2 \qquad \text{and} \qquad \delta_{ij} = 2\zeta_i \omega_i + 2\zeta_j \omega_j \tag{15c}$$

Proof: Defining the ith modal states as $x_i^T = [\eta_i \ \ \dot{\eta}_i]$, it follows from the definition of the cost correlation Ψ_{ij} and the modal outputs ψ_i that

$$\Psi_{ij} = \lim_{t \to \infty} \mathcal{E} \left\{ \frac{1}{t} \int_0^t x_i^T(\sigma) C_i^T C_j x_j(\sigma) \, \mathrm{d}\sigma \right\} = \mathrm{tr}\{ C_i^T C_j X_{ji} \} \tag{16a}$$

where

$$X_{ij} = \lim_{t \to \infty} \mathcal{E} \left\{ \frac{1}{t} \int_0^t x_i(\sigma) x_j^T(\sigma) \, \mathrm{d}\sigma \right\} \tag{16b}$$

$$C_i = [c_i \ \ \underline{c}_i] \tag{16c}$$

Table 1 Model error comparison for example 1 ($\omega_2 = 2$, $\zeta = 0.075$)

	Based on modal costs		Based on model error	
r	Retained modes	Model error	Retained modes	Model error
1	{1}	6.7271e − 03	{1}	6.7271e − 03
2	{1, 3}	3.9368e − 03	{1, 2}	2.9686e − 03
3	{1, 3, 5}	3.8985e − 03	{1, 2, 3}	1.3641e − 04
4	{1, 3, 4, 5}	3.7986e − 03	{1, 2, 3, 4}	5.3301e − 05

Table 2 Model error comparison for example 2 ($\omega_2 = 1$, $\zeta = 0.005$)

	Based on modal costs		Based on model error	
r	Retained modes	Model error	Retained modes	Model error
1	{1}	5.0131e − 01	{1}	5.0131e − 01
2	{1, 3}	4.5809e − 01	{1, 2}	4.5479e − 02
3	{1, 3, 4}	4.5663e − 01	{1, 2, 3}	2.2557e − 03
4	{1, 3, 4, 5}	4.5583e − 01	{1, 2, 3, 4}	7.9952e − 04

The steady-state correlation matrix between the ith and the jth modal states is given by

$$X_{ij} = \begin{bmatrix} X_{ij}^{11} & X_{ij}^{12} \\ X_{ij}^{21} & X_{ij}^{22} \end{bmatrix} \tag{17a}$$

where

$$\begin{cases} X_{ij}^{12} = -X_{ij}^{21} = \dfrac{\Delta_{ij}}{\delta_{ij}} X_{ij}^{11} \\ X_{ij}^{22} = \dfrac{b_i^T b_j}{\delta_{ij}} - \dfrac{\Delta_{ij}}{\delta_{ij}} X_{ij}^{12} \end{cases} \tag{17b}$$

with X_{ij}^{11} as defined in Eq. (15b). Substitution of the Eqs. (17a) and (17b) into Eq. (16a) completes the proof.

Expressions similar to Eqs. (17a) and (17b) can be found in several references (e.g., Ref. 6). Expression (15a) is valid for all i and j, regardless of whether the frequencies are repeated or whether the damping is small. Of course, the expression simplifies under special cases, as shown in the following:

Case 1:

$$\text{If} \quad \omega_i = \omega_j, \quad \text{then} \quad \Psi_{ij} = \frac{b_i^T b_j}{2\omega_i^3(\zeta_i + \zeta_j)} \{c_i^T c_j + w_i^2 \underline{c}_i^T \underline{c}_j\} \tag{18a}$$

Case 2:

$$\text{If} \quad \omega_i \neq \omega_j, \quad \text{and both} \quad \zeta_i, \zeta_j \to 0$$

$$\text{then} \quad \Psi_{ij} \to -\frac{b_i^T b_j}{\Delta_{ij}} \{\underline{c}_i^T c_j - c_i^T \underline{c}_j\} \tag{18b}$$

Case 3:

$$\text{If} \quad \omega_i = \omega_j, \quad \text{and both} \quad \zeta_i, \zeta_j \to 0$$

$$\text{then} \quad \Psi_{ij} \to \frac{b_i^T b_j}{4\zeta_i \omega_i^3} \{c_i^T c_j + \omega_i^2 \underline{c}_i^T \underline{c}_j\} \tag{18c}$$

Note by the special case 2 that, for a lightly damped system (such as a flexible space structure) with distinct frequencies and with either 1) $b_i^T b_j = 0$ for all i and j or 2) $\underline{c}_i^T c_j = 0$ for all i and j, one obtains $\Psi_{ij} \to 0$ for all $j \neq i$, and for all i. Therefore, it follows from Eqs. (8), (9a), (11b), (12b), and (12e) that

$$\hat{\mathcal{V}}_i = \Psi_{ii}, \quad \delta\mathcal{V} = \sum_{i \in N_{\text{tru}}} \Psi_{ii}, \quad \text{and} \quad \delta\mathcal{V} = \widehat{\Delta\mathcal{V}}$$

Hence, for systems satisfying these assumptions, reduced models obtained by MCA, which are guaranteed to minimize the predicted cost error $\widehat{\Delta\mathcal{V}}$, also minimize the model error $\delta\mathcal{V}$.

Examples

Two examples are provided to show that reduced models generated as suggested in this Note will be at least as good as, if not better than, those produced by MCA. In both the examples we begin with five modes and produce reduced models of different orders. The reduced models are evaluated in terms of their model errors, and the results are tabulated in Table 1. The two examples differ only in the frequency of the second mode and in the damping ratios (the damping ratios are assumed to be equal for all of the modes): In example 1, $\omega_2 = 2$ and $\zeta = 0.075$; in example 2, $\omega_2 = 1$ and $\zeta = 0.005$. The parameters common to both the examples are $\omega_1 = 1$, $\omega_3 = 9$, $\omega_4 = 16$, $\omega_5 = 25$, $b_1 = 0.9877$, $b_2 = -0.309$, $b_3 = -0.891$, $b_4 = 0.5878$, $b_5 = 0.707$, $c_1 = 0.9877$, $c_2 = 0.309$, $c_3 = -0.891$, $c_4 = -0.5878$, $c_5 = 0.707$, and $\underline{c}_i = 0$ for all i.

Conclusions

The quality of a reduced model is often computed through the model error. In general, this can be computed only after the reduced model is obtained. This Note points out that, for mechanical systems represented in modal coordinates, the model error can be computed a priori for any choice of retained modes. In fact, the computations involved are just as simple as those needed for the general modal cost analysis. It is shown that, when the system is known to satisfy certain assumptions, the reduced models produced by modal cost analysis, which minimize the predicted cost error, actually minimize the model error. However, this is generally not the case. To facilitate the selection of the set of retained modes that would actually minimize the model error, a cost correlation matrix has been presented. The model error is computed by simple summation of the appropriate elements of this matrix. Although the cost correlation matrix is constructed from the given modal data only once, the computational effort may still become formidable, depending on the number of the original modes and the number of the retained modes.

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